

Note on a theorem of Bangert

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Abstract: We generalize Bangert's non-hyperbolicity result for uniformly tamed almost complex structures on standard symplectic \mathbb{R}^{2n} to asymptotically standard symplectic manifolds.

Keywords: (non-)complex hyperbolicity, asymptotically standard, rationally connected, almost Kähler

MR(2000) Subject Classification: Primary: 58D10 Secondary: 32Q45, 53D45

1. INTRODUCTION

Let M be a smooth manifold and J an almost complex structure on M . Any given $v \in TM$ is tangent to a J -holomorphic disc (see [1], [2]). Therefore we could define Kobayashi-Royden pseudo-norm $\|\cdot\|_{k,J}$ of tangent vectors as:

$$\|v\|_{k,J} = \inf \{R^{-1} : \exists \text{ a pseudo-holomorphic } f : B_R \rightarrow (M, J), df(\partial_x) = v\}$$

Here $B_R \subset \mathbb{C}$ is the ball of radius R , ∂_x the unit tangent vector along the real axis. This in turn induces a pseudo-distance on M , which is exactly the Kobayashi (pseudo-)distance. If such an induced distance is a metric, then J is called (Kobayashi-)hyperbolic. Another well-known notion of complex hyperbolicity is *Brody hyperbolicity*, which means the absence of holomorphic lines in (M, J) . These two notions are equivalent on compact manifolds, see [3].

In a non-compact Riemannian manifold (M, g) , it seems natural to consider the uniform property of hyperbolicity. We say J is *(g-)uniformly hyperbolic*, if $\|v\|_g \leq C \cdot \|v\|_{k,J}$, for all $v \in TM$ and a constant C independent of v . This is equivalent to saying that the set

$$\{R \mid \exists \text{ a pseudo-holomorphic map } f : B_R \rightarrow (M, J), \|f'(0)\|_g = 1\}$$

is bounded. In particular, it is easy to see that for a non-compact complete manifold, the completeness of Kobayashi distance (see [1], [4], [5]) follows from the uniform hyperbolicity. If M is non-compact and (M, J) is not Brody hyperbolic, that is, there exist a complex line, then for any Riemannian metric g , J cannot be uniformly hyperbolic.

The notion of uniform hyperbolicity was studied first by V. Bangert for the class of J uniformly tamed by standard symplectic form ω_0 on $(\mathbb{R}^{2n}, g_{eud})$ in [6], where he proved that none of them is uniformly hyperbolic. Recall that when (M, ω) is a symplectic manifold, an almost-complex structure J on M satisfying $\omega(v, Jv) > 0$ for any tangent vector $v \in TM$ is called ω -tamed (see [7]), while the notion *uniformly tamed* introduced by Bangert in [6] is more suitable in the setting that

^{0*} The first author is supported by NSF grant.

(M, ω) is a non-compact symplectic manifold endowed with a preferred metric g (see Definition 2.1). Motivated by [6], in this note we generalize Bangert's result to asymptotically standard symplectic manifolds.

Definition 1.1. A (non-compact) symplectic manifold with Riemannian metric (M, g, ω) is *asymptotically standard* if for some compact subset K in M , there is a map $\phi_M : (M \setminus K, g, \omega) \rightarrow (\mathbb{R}^{2n} \setminus B_{\bar{R}}, g_{eud}, \omega_0)$ which is both a symplectomorphism and an isometry.

It should be noted that in the case when (M, ω) is symplectic aspherical, it is proved by Eliashberg, Floer and McDuff that M is diffeomorphic to \mathbb{R}^{2n} (cf. [8],[9]). Our main result in this paper reads as follows:

Theorem 1.2. *For an asymptotically standard symplectic manifold (M, g, ω) , any uniformly tamed almost complex structure is not uniformly hyperbolic.*

In essence what we need is to construct a sequence of holomorphic disks with radius approaching infinity, while the norm of the differential at the origin remains 1. The starting point of the proof is the “almost-Kähler cut” technique due to D. Burns, V. Guillemin, and E. Lerman in [10], which provides a convenient way to compactify the manifold while keeping track of the almost complex structure. By [11] (see also [12]), the compactification is easily seen to be rationally connected. We thus obtain a sequence of holomorphic disks inside larger and larger parts of M . One then uses the reparametrization process developed in [6] to obtain needed estimates for derivatives.

There are a few remarks we need to make about Theorem 1.2. Similar problems have been explored by various authors, see [13], [14]. Especially, when (ω, J) is a compatible pair, uniform hyperbolicity is equivalent to the notion of *almost-Kähler hyperbolicity* in [13] with g being the induced metric by the (ω, J) . A notion of Floer theoretical symplectic hyperbolicity was introduced in [13], which is quite different, though related. Bangert's result is then extended by Biolley to many generalized Stein manifolds in the class of compatible almost-complex structures, with some restrictions on capacity.

It is notable that similar patterns appear in the far-reaching symplectic field theory, see [16], [15], [17], [18], [19], etc. The “almost Kähler cut” technique we employ in this paper should be viewed somehow as the process of “stretching the neck” in symplectic field theory, with the addition feature of retaining the almost complex structure.

Acknowledgement: The authors would like to thank Shinichiroh Matsuo for suggesting the paper of V. Bangert, and helpful discussions about results in his upcoming papers on analogous problems in gauge theory settings. Gratitude is also due to Jianxun Hu and Guangcun Lu for inspiring conversations. Special thanks to the anonymous referees who gave numerous suggestions making the exposition of this note much more clear. The second author would also like to thank Josef Dorfmeister, Guoyi Xu and Weiyi Zhang for their interests in this work.

2. UNIFORMLY TAMED ALMOST COMPLEX STRUCTURES

Definition 2.1. Let (M, g) be a Riemannian manifold equipped with a symplectic form ω , and an almost complex structure J over M . We say J and ω are *(g-)uniformly tamed* by each other if:

- (1) (uniform boundedness of ω) For some $\alpha > 0$, $\omega(u, v) \leq \alpha \|u\|_g \|v\|_g$
- (2) (uniform boundedness of J) For some $\beta > 0$, $\|Jv\|_g \leq \beta \|v\|_g$

(3) (uniform tameness) For some $c > 0$, $\omega(v, Jv) \geq c\|v\|_g^2$

Remark 2.2. From the definition above, one easily deduce that in the case of uniformly tamed pair, the metric defined by $\tilde{g}(u, v) = \frac{1}{2}(\omega(u, Jv) - \omega(Ju, v))$ is equivalent to the metric g , in that there is a constant $a > 0$ satisfying $\frac{1}{a}\|v\|_{\tilde{g}} \leq \|v\|_g \leq a\|v\|_{\tilde{g}}$ for all tangent vector v . Moreover, a J -holomorphic curve has the same symplectic area as its associated \tilde{g} -area. This follows from that the pull-back metric has volume form $\omega((f_R)_*(v), J(f_R)_*(v))ds \wedge dt$, where ds, dt are dual to v, iv , respectively (see also the proof of Lemma 1.2 in [6]). In what follows we will call β the *bounding constant* and c the *taming constant*.

Now we focus on almost-complex structures on \mathbb{R}^{2n} . Let $\mathcal{J}(\omega_0)$ denote the space of complex structures tamed by ω_0 , and $\mathcal{J}_{\beta, c}$ those uniformly tamed by ω_0 with bounding constant β and taming constant c . Following [6], note that $\mathcal{J}_{\beta, c}$ is closed and bounded as a subset of $\mathcal{J}(\omega_0)$, therefore compact. Since the tangent bundle of \mathbb{R}^{2n} is trivial, a uniformly tamed almost-complex structure $J_{\beta, c}$ with taming constant β and bounding constant c on a subset S of \mathbb{R}^{2n} yields a map from S to $\mathcal{J}_{\beta, c}$. Conversely, a compact subset of $\mathcal{J}(\omega_0)$ is bounded with a uniformly tamed constant. Therefore,

Lemma 2.3. *A uniformly tamed almost-complex structure $J_{\beta, c}$ over $S \subset \mathbb{R}^{2n}$ is equivalent to a map with precompact image $f : S \rightarrow \mathcal{J}_{\beta, c}$.*

3. A VARIATION OF ALMOST KÄHLER CUT

3.1. Almost Kähler cut. We now give a brief review of the technique of almost-Kähler cuts. Readers are referred to [10] for a more detailed exposition on the subject. It should be noted that although [10] concerns with Kähler manifolds only, the results we use in this note can be easily (even directly) adapted to the context of almost-Kähler manifolds.

We first review the almost-Kähler reduction process. The reduction of a Kähler manifold is well-known ([20]). The reduction of an almost-Kähler manifold is parallel and even simpler since we do not need to check the integrability of J , thus omitting the concrete construction of the Levi-Civita connection.

Assume (M, ω, J, g) is an almost-Kähler manifold with a free S^1 -Hamiltonian action also preserving J (hence preserving g as well). Let X be the generating vector field of the S^1 -action. Suppose μ is a moment map. Then

$$\nabla \mu = JX.$$

It follows that each tangent space of $\mu^{-1}(0)/S^1$ has a quotient complex vector space structure. Therefore $\mu^{-1}(0)/S^1$ has an induced almost-Kähler structure.

Let (M, ω, J) be an almost-Kähler manifold with a Hamiltonian S^1 -action $\tau : S^1 \times M \rightarrow M$, ϕ the corresponding moment map, \mathbb{C} with its standard Kähler structure. Define an S^1 -action

$$(3.1) \quad \begin{aligned} \tilde{\tau} : S^1 \times (M \times \mathbb{C}) &\rightarrow M \times \mathbb{C} \\ (e^{i\theta}, (x, z)) &\rightarrow (e^{i\theta}x, e^{i\theta}z) \end{aligned}$$

with moment map $\Psi(x, z) = \phi(x) + \|z\|^2$.

Definition 3.1. We call the almost-Kähler manifold $\Psi^{-1}(\lambda)/S^1$ the *almost-Kähler cut of (M, ω, J) along level set λ* , denoted as M_λ .

Since the tangent space of $D_\lambda = \{(x, z) \in M_\lambda | \phi(x) = \lambda\}/S^1$ is naturally identified with that of reduced space $\phi^{-1}(\lambda)/S^1$, D_λ is an almost complex hypersurface of M_λ . Denote

$$(3.2) \quad M'_\lambda = M_\lambda - D_\lambda.$$

Consider also the open submanifold of M ,

$$M_0^\lambda = \{x \in M | \phi(x) < \lambda\}$$

M'_λ and M_0^λ come with induced almost complex structure in the obvious way. For example, the former with the quotient almost-Kähler structure, and the latter as a subset of M . The relation between symplectic structures of these manifolds is clear. As pointed out by E. Lerman [21],

$$(3.3) \quad \begin{array}{ccc} T^{-1} : & M_0^\lambda & \rightarrow & M'_\lambda \\ & x & \mapsto & (x, \sqrt{\lambda - \phi(x)}) \end{array}$$

is actually a symplectomorphism. However, T is not an almost complex isomorphism between the two manifolds with the almost complex structures specified above.

Suppose the S^1 -action on (M, ω, J) extends to an almost complex action of \mathbb{C}^* still denoted by τ . For $z \in \mathbb{C}^*$, let τ_z be the corresponding almost complex isomorphism of M . As observed in [10], τ_{e^t} is generated by $\nabla\phi$. Let

$$(3.4) \quad \iota_0 : M_0^\lambda \rightarrow M, \quad \iota_0(p) = \tau_{\frac{1}{\sqrt{\lambda - \phi(p)}}}(p)$$

The central result in [10] we are going to use is the following:

Theorem 3.2. ([10], Theorem 2.1) With notations above, then

$$(3.5) \quad \iota_0 T : M'_\lambda \rightarrow \iota_0(M_0^\lambda)$$

is an almost complex isomorphism.

Roughly speaking, this theorem asserts that an open part of the almost-Kähler cut is biholomorphic to the open set of M which is the union of M_0^λ and the unstable manifold starting therein. For readers' convenience, we include a sketch of the proof in [10] with adaption to the almost-Kähler case. Consider the biholomorphic map

$$f : M \times \mathbb{C}^* \rightarrow M \times \mathbb{C}^*$$

defined by $f(p, z_0) = (\tau_{z_0} p, z_0)$. Consider actions R and D of \mathbb{C}^* on $M \times \mathbb{C}^*$ defined as

$$R(z, (p, z_0)) = (p, z z_0),$$

and the diagonal action:

$$D(z, (p, z_0)) = (\tau_z p, z z_0),$$

then f intertwines R and D , i.e. $f \circ R(z, -) = D(z, f(-))$. Therefore, the pullback by f of the natural product symplectic structure is invariant under the S^1 -action on the \mathbb{C}^* factor (coming from the \mathbb{C}^* -action restricted to the unit circle) with moment map

$$\tilde{\Psi}(p, z) = \phi(\tau_z p) + |z|^2$$

Here f maps the level set of $\tilde{\Psi}^{-1}(\lambda)$ onto the level set $\Psi^{-1}(\lambda)$ and induces an almost-Kähler isomorphism:

$$h : \tilde{\Psi}^{-1}(\lambda)/S^1 \rightarrow \Psi^{-1}(\lambda)/S^1$$

Note that the right-hand-side is identified with the almost-Kähler cut as desired. Now we identify the left-hand-side of our isomorphism with the original manifold and h with $(\iota_0 \circ T)^{-1}$ as follows: remember we removed the origin of \mathbb{C} in the product manifold, therefore, the S^1 -quotients are actually free without cut-divisor. Thus we have the following identification, where we identify the quotient of \mathbb{C}^* by S^1 with the positive real ray:

$$\tilde{\Psi}^{-1}(\lambda)/S^1 = \{(p, e^t) | \phi(\tau_{e^t} p) + e^{2t} = \lambda\}$$

and,

$$\Psi^{-1}(\lambda)/S^1 = \{(p, e^t) | \phi(p) + e^{2t} = \lambda\}$$

For $\tilde{\Psi}^{-1}(\lambda)/S^1$, notice that as an almost-complex manifold, it is $M \times \mathbb{C}^*/\mathbb{C}^*$, identified with $\iota_0(M_0^\lambda)$ with restricted almost-complex structure from M (since f is a biholomorphism, what we did on the complex structure is simply multiplying M by a \mathbb{C}^* factor and then quotienting it out). Under these identifications and (3.3), $h(p, e^t) = (\tau_{e^t} p, e^t)$, hence

$$h^{-1}(p, e^t) = (\tau_{\sqrt{\lambda - \phi(p)}}^{-1}(p), \sqrt{\lambda - \phi(p)})$$

Hence $h = (\iota_0 \circ T)^{-1}$, and in particular, the domain of h is $\iota_0(M_0^\lambda)$. Note that T is just a step of identification which drops or picks up the second factor. This proves the theorem. For more details one is referred to [10].

Remark 3.3. We adopted different notations in Theorem 3.2 from what is in [10]. In this note we use the notation M'_λ in place of the almost-Kähler manifold “ M_0^λ with its M^λ complex structure” just for convenience. The content of the statement is completely the same as it is in [10].

Example. The simplest example of Kähler cut is to consider the symplectic cut of \mathbb{C} with standard Kähler structure at $\|z\| = r$. This induces a symplectomorphism from $\{\|z\| < r\}$ to $\mathbb{C}P^1 \setminus \{pt\}$ with symplectic form $\frac{r^2}{\pi} \omega_{FS}$. On the other hand, stereographic projection gives a biholomorphic map between \mathbb{C} and $\mathbb{C}P^1 \setminus \{pt\}$, while these structures give a Kähler structure in the cut space. Theorem 3.2 gives a formula between the moment map of S^1 -action and the biholomorphism, which could be viewed as a generalization of stereoprojection. Note that the same holds for \mathbb{C}^n , and after the cut at $\|z\| = r$, the symplectic area of the generator of $H^2(\mathbb{C}P^n)$ is r^2 .

3.2. A variation. In this subsection, (M, ω) is a symplectic manifold with a preferred metric g .

Definition 3.4. (Extendable S^1 -action) Suppose J is an almost complex structure on (M, ω, g) . A Hamiltonian S^1 -action τ on M with a moment map ϕ is called J -extendable if it is extended to an almost complex semi-group action of the complement of open unit disk in \mathbb{C} , $\tilde{\tau} : \mathbb{C} \setminus D_1 \times M \rightarrow M$, such that the gradient flow of the moment map ϕ generates the infinitesimal action of $\tilde{\tau}_{e^t}$. Here S^1 is identified with the unit circle of \mathbb{C} .

(M, ω, J) is called *asymptotically almost-Kähler (AAK)* if there is a compact subset E , where $(M \setminus E, \omega, J)$ is an almost-Kähler manifold. To avoid possible confusion, an extendable S^1 -action on an AAK manifold always acts only on $M \setminus E$, and the reference metric is taken to be the one induced by (ω, J) . It is easy to see that asymptotically standard manifolds satisfies all assumptions above: (see also [10])

Lemma 3.5. *An asymptotically standard manifold is AAK and the standard rotation is an extendable S^1 -action.*

Remark 3.6. *This lemma is still valid if (M, ω, J) is asymptotically of the form $(\mathbb{R}^{2n}, \omega = -\sqrt{-1}\partial\bar{\partial}F(\|z\|^2), J_0)$, where F is a real valued function such that $\frac{1}{\epsilon} > F'(x) + xF''(x) \geq \epsilon > 0$ is bounded.*

For our purpose, we need an adapted version of Theorem 3.2 by allowing an extendable S^1 -action on only the non-compact part of the AAK manifold. We first fix some notations for the convenience of exposition. In the rest of the note, we make the convention that $=_d, =_s, =_h, =_k$ denotes diffeomorphism, symplectomorphism, biholomorphism and almost-Kählerian isomorphism, respectively. We also use the following notation:

$$(3.6) \quad M_I = \{x \in M \mid \phi(x) \in I \subset \mathbb{R}\}$$

With the above preparation, we are ready to state our lemma:

Lemma 3.7. *Let (M, ω, J) be an AAK manifold with an extendable S^1 -action on the complement of a compact set $\tau : S^1 \times M \setminus K \rightarrow M \setminus K$. Then there is an ω_{red} -tamed almost complex structure J_{red} on M_λ , such that*

$$(M'_\lambda, J_{red}) =_h (M, J)$$

Proof. Let $\phi : M \setminus K \rightarrow \mathbb{R}$ be the moment map on the asymptotic part of M . Suppose the S^1 -action is extendable on $\phi \geq \lambda_0$, choose $\lambda \gg \lambda_0$, let $\Psi : M_{\geq \lambda_0} \times \mathbb{C} \rightarrow \mathbb{R}$ be the moment map as in the definition of Kähler cut in section 2. Decompose

$$\begin{aligned} \Psi^{-1}(\lambda) &= \{(x, z) \mid \phi(x) + |z|^2 = \lambda\} \\ (3.7) \quad &= \{(x, z) \mid \phi(x) \leq \lambda - 1\} \cup \{(x, z) \mid \lambda > \phi(x) > \lambda - 1\} \cup \{(x, z) \mid \phi(x) = \lambda\} \\ &= A_1 \cup A_2 \cup A_3 \end{aligned}$$

and $\Psi^{-1}(\lambda)/S^1$ has a natural symplectic form ω_{red} from symplectic reduction.

Note that ι_0 itself is identity on $\phi = \lambda - 1$, to define an ω_{red} -tamed almost complex structure on M_λ , we first induce an almost complex structure $T_*^{-1}(J)$ on A_1/S^1 by restricting the symplectomorphism (3.3) to level set below $\lambda - 1$. For the non-compact part, we denote $N_{[\lambda-1, \lambda]}$ to be the reduction of

$$(3.8) \quad M_{[\lambda-1, \lambda]} = \{x \in M \setminus K \mid \lambda - 1 \leq \phi(x) \leq \lambda\}$$

Moreover, by the assumption of λ , the reduction is almost-Kählerian. In particular, by Theorem 3.2 and (3.3), the reduced space would have the following property:

$$(3.9) \quad (N_{[\lambda-1, \lambda]}, J_{red}) =_h (M_{[\lambda-1, \infty)}, J)$$

$$(3.10) \quad (N_{[\lambda-1, \lambda]}, \omega_{red}) =_s (M_{[\lambda-1, \infty)}, \omega)$$

where the biholomorphism in (3.9) is induced by $(\iota_0 T)^{-1}|_{\mathbb{R}_{[\lambda-1, \infty)}^{2n}}$ and the symplectomorphism in (3.10) by $T^{-1}|_{M_{[\lambda-1, \lambda]}}$. Since the reduced space is an almost-Kähler manifold, *a priori* we know the pair on the left hand side of both equations above will consist of an almost-Kähler pair. Therefore, we have almost-Kähler isomorphism via T^{-1} :

$$(M_{[\lambda-1, \lambda]}, \omega, (\iota_0)_*^{-1}(J)) =_k (N_{[\lambda-1, \lambda]}, \omega_{red}, J_{red})$$

In sum, we induce an ω_{red} -tamed almost complex structure \tilde{J} on $(M_\lambda, \omega_{red})$ using the C^1 -diffeomorphism:

$$(3.11) \quad F = \iota_0 \# id : M_{<\lambda} \rightarrow M$$

$$\iota_0 \# id = \begin{cases} \iota_0, & \phi(x) \geq \lambda - 1 \\ id, & \phi(x) \leq \lambda - 1 \end{cases}$$

To get a smooth almost-complex structure, one only needs to perturb F and use the openness of uniformly tamed almost-complex structure with taming constant less than a fixed $\beta > 0$. In the following such a choice of perturbation would not matter so we don't need to specify our choice of perturbation. Thus \tilde{J} is the tamed almost-complex structure as required. \square

Remark 3.8. This diffeomorphism is in fact the composition of T and a time-1 map of an integral flow of a continuous vector field on M , see [10] for a flow representation of ι_0 . Note that in the special case when M is asymptotically standard, the taming and bounding constant are unchanged when almost-Kähler cut is performed along a standard sphere $M_\lambda \subset \mathbb{R}^{2n}$. The reason is that the pair are unchanged in the compact set $M_{<\lambda}$; outside it is a subset of Euclidean space before the cut and complex projective space after the cut, with all corresponding structures, therefore the taming constant remains to be 1.

4. REVIEW OF BANGERT'S RESULTS

In this section we collect definitions and facts in [6] for the proof of Theorem 1.2, and we refer the reader to proofs therein for details. We adopt standard notations in [22]. For example, if S is an m -current in (M, g) , $\mathbf{M}(S)$ denotes its mass. When S is rectifiable, it is representable by integration and can be written as $S = \|S\| \wedge \vec{S}$. Here $\|S\|$ is a canonically associated Borel measure on M finite on compact sets, and $\vec{S} : M \rightarrow \bigwedge^m M$ is the unit vector field orienting the associated set of S . If we have moreover a Borel subset B on M , the *interior multiplication* $S \llcorner B$ is the unique current represented as $S \llcorner B = \chi_B \|S\| \wedge \vec{S}$, where χ_B is the characteristic function of B . For a proper Lipschitz map f and a rectifiable current S , the push-forward current is denoted as $f_{\#}(S)$, which is also rectifiable. See [22] or [23] for the precise definitions. Next we recall the notion of quasi-minimality:

Definition 4.1. Let (M, g) be a Riemannian manifold and $Q > 1$. A rectifiable current S in M is called quasiminimizing with constant Q , or simply Q -minimizing if

$$\mathbf{M}(S \llcorner B) \leq Q \mathbf{M}(X)$$

when B is a Borel subset of M and X a rectifiable current in M satisfying $\partial X = \partial(S \llcorner B)$

Lemma 4.2. Let ω be an exact symplectic form, J an almost complex structure on (M, g) . Suppose they form a uniformly tamed pair. Let (S, j) be a compact Riemann surface with orientation induced by complex structure j , and $f : (S, j) \rightarrow (M, J)$ a pseudo-holomorphic map. Then the rectifiable current $f_{\#}(S)$ is quasi-minimizing with $Q = \alpha\beta/C$, with the notations in Definition 2.1.

Now a very useful lemma reads:

Lemma 4.3. For all real constants $Q > 1, t > 0$, and integers $k \geq 1$, there exist $c = c(Q, t, k) \in (0, 1)$ such that the following is true for all $R > 0$: If S is a Q -minimizing rectifiable k -current in Euclidean \mathbb{R}^n s.t.

$$\text{supp}(\partial S) \subseteq B(cR), \mathbf{M}(S) \leq tR^k,$$

then

$$\text{supp}(S) \subseteq B(R).$$

□

We could adapt Lemma 4.3 to asymptotically standard manifold M for J -holomorphic curves in our case. For convenience, in the rest of the paper we denote the closure of $(\phi_M^{-1}(\mathbb{R}^{2n} \setminus B_R))^c \subset M$ by V_R , $R > \bar{R}$ (See Definition 1.1 for notations).

Suppose J is a given uniformly tamed almost-complex structure, then J restricts to the non-compact standard part of M , therefore extends to give an almost-complex structure J' of $(\mathbb{R}^{2n}, \omega_0)$ via ϕ_M . By Lemma 2.3 and contractibility of $\mathcal{J}(\omega_0)$, we could choose such extension by contracting $\mathcal{J}_{\beta, c}$ to J_{std} , so that the almost-complex structure on the sphere ∂V_R extends to the ball B_R . Since such contraction can be chosen once and for all and the trace is compact, we could assume the extended J' is uniformly tamed with taming and bounding constants independent of R . Moreover, we thus obtain a constant $c'(Q, J', t) > 0$ from Lemma 4.3, such that for any J' -holomorphic curves

S' with $\text{supp}(\partial S') \subseteq B(c'R)$, $M(S') \leq tR^2$, one has $S' \subset B_R \subset \mathbb{R}^{2n}$. Now for any $R > \frac{1}{c'}\bar{R}$, let S be a J -holomorphic curve with boundary in $V_{c'R}$, then S is divided into $S \cap V_{c'R}$ and $S \setminus V_{c'R}$. We could apply Lemma 4.2 and Lemma 4.3 to $S \setminus V_{c'R}$ by identifying $S \setminus V_{c'R}$ with J -holomorphic curve with boundary on $V_{c'R}$ in \mathbb{R}^{2n} . Therefore,

Lemma 4.4. *Lemma 4.3 is true for J -holomorphic curves in asymptotically standard manifolds (M, ω, g) when J is given, uniformly tamed by ω , and $c'R > \bar{R}$.*

Remark 4.5. Lemma 4.4 in particular shows an asymptotic standard manifold M with a uniformly tamed almost complex structure is a *tame almost complex manifold*, in the sense of Sikorav [24], which asserts that for any compact set $K \subset M$ and positive class C , there is another compact subset $K' \subset M$, such that every J -holomorphic curve in the class C intersecting K is contained in K' . See [13] for another generalization of Lemma 4.3 in Stein manifolds.

To deal with non-compact target space, Bangert proved the following limiting process holds:

Proposition 4.6. *Let (M, g, J) be an almost-complex manifold with a J -invariant metric g . Suppose there is a sequence $R_j \rightarrow \infty$, constant $C > 0$, J -holomorphic maps $h_j : D_j \rightarrow M$ defined on topological disks $D_j \subset \mathbb{C}$, where $h_j(0)$ lies in a compact subset $K \subset M$, such that*

- (1) $\text{dist}_g(h_j(0), h_j(\partial D_j)) \geq \alpha R_j$ and
- (2) $\text{area}_{g_0}(h_j) \leq CR_j^2$

Then (M, g, J) is not uniformly hyperbolic.

With the aid of this proposition, the problem is reduced to finding such D_j 's. Note that Bangert did not state this proposition in full generality, but the proof is easily seen to be valid (cf. [6], pp. 39, Proof of Proposition 2.7).

5. PROOF OF THEOREM 1.2

Proof. We follow Bangert's idea to construct holomorphic disks as is required in Proposition 4.6. Let J_{std} denote the standard complex structure on \mathbb{R}^{2n} or $\mathbb{C}P^n$, depending on the space we are talking about. We also use notations in Section 4.

Now for any $R > \bar{R}$, and fixed J uniformly tamed by ω in the metric g_0 , with bounding constant β and taming constant c , we can find J_R , such that the following holds:

- (1) $J_R(x) = J_{std}(x)$, if $x \notin V_{2R}$ and $J_R = J$ if $x \in V_R$.
- (2) J_R is uniformly bounded and uniformly tamed by ω with taming constant $\beta'(\beta, c)$ and bounding constant $c'(\beta, c)$ independent of R .

This catenation is essentially proved in [6]. Indeed, we have a map $T : N(\partial V_R) \cup \mathbb{R}^{2n} \setminus B_{2R} \rightarrow \mathcal{J}(\omega_0)$, with $T(N(\partial V_R)) \subset \mathcal{J}_{\beta, c}$ and $T(\mathbb{R}^{2n} \setminus B_{2R}) = J_{std}$, where $N(\partial V_R)$ is a neighborhood of ∂V_R . As in the proof of Lemma 4.4, we extend this map smoothly by contraction of $\mathcal{J}_{\beta, c}$ to J_{std} to obtain J_R . Again since the catenation is constrained in a compact subset, that is, the trace of the contraction, J_R is a uniformly tamed almost-complex structure with taming and bounding constants independent of R .

For a fixed J_R as above, we could perform almost-Kähler cut at ∂V_{3R} , resulting in a new almost-Kähler manifold $(M'_R, \omega_{red}, J_{red})$, which is almost-complex isomorphic to (M, J_R) by Lemma 3.7,

and preserves the taming constant by Remark 3.8. It is readily seen that a neighborhood of cut-divisor in M_R is identified with a neighborhood of a divisor $\mathbb{C}P^{n-1} \subset (\mathbb{C}P^n, \omega_{FS}, J_{std})$ as an open Kähler manifold. Therefore the two-point Gromov-Witten invariant of M_R in the class $t \in H_2(M_R, \mathbb{Z})$, the dual of the cut-divisor, is non-zero by Theorem 5.1.2 in [11]. Thus there is a stable curve C passing through a fixed point $x \in K$ in class t . Consider the components $\{C_i\}_{i=1}^l$ of C .

Claim. There is exactly one component of C which intersects the cut-divisor, and the intersection is a unique point. Moreover, such a component intersects K as well.

Proof. If a component C_i intersects the cut-divisor, it either intersects positively with or lies entirely in the cut-divisor $(\mathbb{C}P^{n-1}, J_{std})$. In the latter case, the intersection number of C_i and the cut divisor is also positive as the normal bundle of the cut divisor is positive. Since the intersection number of C and the cut divisor is 1, the first two assertions follow.

The last assertion is proved if we can show that it is impossible for any component to lie entirely in $M_R \setminus K$. This is because that $M_R \setminus K$ is a disk bundle over the cut divisor, the class of any component in $M_R \setminus K$ must be a positive multiple of t . Since C also passes through $x \in K$, there must exist other non-constant components. But these components have non-positive total symplectic area, which is impossible. \square

We consider this unique component intersecting the cut-divisor. This holomorphic sphere gives a J_R -holomorphic line f_R in M with $f_R(0) \in K$ and $f_R(\infty)$ the intersection with cut-divisor by a reparametrization. Let $L_R := f_R(\mathbb{C}) \cap \partial V_{R/2}$, and by Sard's theorem, we may assume $f_R^{-1}(L_R)$ is a collection of embedded circles up to an arbitrary small change of R . We choose the largest topological disk $0 \in D_R \subseteq \mathbb{C}$ with boundary on $f_R^{-1}(L_R)$. It is clear from the claim above that the energy, or equivalently, the symplectic area of $S_R := (f_R)_\#([D_R])$ is at most equal to that of the cut-divisor, which is $\tilde{\nu}R^2$, $\tilde{\nu}$ an absolute constant. Note that the area of S_R is bounded by the \tilde{g} -area induced by (ω, J) , up to a constant $\bar{\nu}(\beta, C)$. Therefore Remark 2.2 yields:

$$(5.1) \quad \mathbf{M}(S_R) \leq \nu R^2$$

where $\nu = \nu(\beta, C)$.

The general procedure above does not give holomorphic disks with respect to the almost complex structure J because S_R might touch the “shell” $V_{2R} \setminus V_R$ where the almost complex structure J_R does not coincide with J . To get a genuine J -holomorphic disk, consider a subset of S_R . We have the uniform quasi-minimality constant Q and ν satisfying (5.1), Lemma 4.4 thus gives a uniform constant $c \in (0, 1)$, such that when $f(\partial D') \in V_{cR}$, we have $f(D') \in V_R$. Hence by considering the intersection $\partial V(cR) \cap S_R$, and repeating the above argument with possibly one more application of Sard's theorem, we have a holomorphic disk \bar{D}_R lying entirely in V_R , therefore a J -holomorphic disk, with boundary on V_{cR} . It follows then from Proposition 4.6 that (M, g, J) is not uniformly hyperbolic. \square

Remark 5.1. We in fact did not use Lemma 3.7 in an essential way in the proof above. Instead, one could just use stereographic projection to assign complex structure on the cut space to argue for uniform tameness. However, our proof is generalized in a straightforward way to asymptotic \mathbb{R}^{2n}

with standard metric and the symplectic form in Remark 3.6. One easily see that the neighborhood of cut-divisor as a complex manifold is still $\mathcal{O}(1)$ therefore positive. Moreover, we have also the estimate (5.1). This is equivalent to an estimate of the symplectic area, thus the associated \tilde{g} -area of the cut-divisor. We could view the cut-divisor as $\mathbb{C}P^{n-1}$ with corresponding symplectic form ω , and by a homology argument we reduce the problem to the estimate of the line class pairing with ω . Therefore we could just take the holomorphic disk in x_1x_2 -plane, and by assumption on F , the associated $\tilde{g} = (F' + xF'') \cdot g_{eud}$ is bounded and equivalent to g_{eud} , as desired.

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